Soustavy obyčejných diferenciálních rovnic
METODA VARIACE KONSTANT
Řešené příklady

1.
$$\begin{aligned} x' &= x + 2y + \mathrm{e}^t \\ y' &= 3x + 2y \end{aligned}$$

2.
$$\begin{aligned} x' &= x - 2y \\ y' &= x - y + 1 \end{aligned}$$

3.
$$\begin{aligned} x' &= 2x - y + 8\mathrm{e}^{-2t} \\ y' &= x + 4y + 2\mathrm{e}^{-2t} \end{aligned}$$

(1) $x' = x + 2y + e^t$
$y' = 3x + 2y$

a) $x' = x+2y$
$y' = 3x+2y$

$\begin{vmatrix} 1-r & 2 \\ 3 & 2-r \end{vmatrix} = 0$

$(1-r)(2-r)-6=0$
$r^2-3r-4=0$
$(r+1)(r-4)=0$

$r_1=-1$
$r_2=4$

$r_1=-1: \left(\begin{array}{cc|c} 2 & 2 & 0 \\ \cancel{-3} & \cancel{-3} & \cancel{0} \end{array}\right)$ $2v_1+2v_2=0$
$\bar{v}=(1,-1)$ 1. řešení $\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (*)

$r_2=4: \left(\begin{array}{cc|c} -3 & 2 & 0 \\ \cancel{-3} & \cancel{-2} & \cancel{0} \end{array}\right)$ $-3v_1+2v_2=0$
$\bar{v}=(2,3)$ 2. řešení $\begin{pmatrix} x \\ y \end{pmatrix} = e^{4t}\cdot\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (**)

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\cdot e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\cdot e^{4t}\cdot\begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad c_{1,2}\in\mathbb{R}$$

$x = c_1e^{-t} + 2c_2e^{4t}$
$y = -c_1e^{-t} + 3c_2e^{4t}$

b) <u>variace konstant</u> $x = c_1(t)\cdot e^{-t} + 2c_2(t)\cdot e^{4t}$
$y = -c_1(t)e^{-t} + 3c_2(t)e^{4t}$ (♡)

$$\begin{array}{cc} (*) & (**) \end{array}$$
$$\begin{pmatrix} e^{-t} & 2e^{4t} \\ -e^{-t} & 3e^{4t} \end{pmatrix}\cdot\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$

Cramerovo pravidlo: $\begin{vmatrix} e^{-t} & 2e^{4t} \\ -e^{-t} & 3e^{4t} \end{vmatrix} = 3e^{-t}\cdot e^{4t} + 2e^{-t}\cdot e^{4t} = 5e^{3t} \neq 0$

$$c_1'(t) = \frac{\begin{vmatrix} e^t & 2e^{4t} \\ 0 & 3e^{4t} \end{vmatrix}}{5e^{3t}} = \frac{3e^{5t}}{5e^{3t}} = \frac{3e^{2t}}{5}$$

$$c_2'(t) = \frac{\begin{vmatrix} e^{-t} & e^{t} \\ -e^{-t} & 0 \end{vmatrix}}{5e^{3t}} = \frac{1}{5e^{3t}} = \frac{e^{-3t}}{5}$$

$$c_1(t) = \int c_1'(t)\cdot dt = \int \frac{3e^{2t}}{5}\cdot dt = \frac{3e^{2t}}{10}$$

$$c_2(t) = \int c_2'(t)\cdot dt = \int \frac{e^{-3t}}{5}\cdot dt = \frac{e^{-3t}}{-15}$$

$(\heartsuit)$: $x = c_1(t)\cdot e^{-t} + 2c_2(t)e^{4t} = \frac{3e^{2t}}{10}\cdot e^{-t} + 2\cdot\frac{e^{-3t}}{-15}\cdot e^{4t} = \frac{e^{t}}{6}$

$y = -c_1(t)e^{-t} + 3c_2(t)e^{4t} = -\frac{3e^{2t}}{10}\cdot e^{-t} + 3\cdot\frac{e^{-3t}}{-15}\cdot e^{4t} = -\frac{e^{t}}{2}$

c) $\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2\begin{pmatrix} 2e^{4t} \\ 3e^{4t} \end{pmatrix} + \begin{pmatrix} \frac{e^{t}}{6} \\ -\frac{e^{t}}{2} \end{pmatrix}$ $\quad c_{1,2}\in\mathbb{R}$

$$x = c_1e^{-t} + 2c_2e^{4t} + \frac{e^{t}}{6}$$
$$y = -c_1e^{-t} + 3c_2e^{4t} - \frac{e^{t}}{2}$$
$c_{1,2}\in\mathbb{R}$

② $x' = x - 2y$

$y' = x - y + 1$

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a) $\begin{vmatrix} 1-r & -2 \\ 1 & -1-r \end{vmatrix} = 0$ $\quad (1-r)(-1-r)+2=0$

$r^2+1=0 \quad \Rightarrow \quad r_{1,2} = \pm i$

$r_1 = i: \quad \left(\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right) \cdot(1+i) \sim \left(\begin{array}{cc|c} \cancel{2} & \cancel{-2-i} & \cancel{0} \\ 1 & -1-i & 0 \end{array}\right)$ $\quad v_1 + (-1-i)v_2 = 0$

$\vec{v} = (1+i, 1)$

$$\begin{pmatrix} 1+i \\ 1 \end{pmatrix} \cdot e^{it} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \cdot (\cos t + i \cdot \sin t) = \begin{pmatrix} \cos t + i \cdot \sin t + i \cdot \cos t - \sin t \\ \cos t + i \cdot \sin t \end{pmatrix} =$$

$$= \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + i \cdot \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \cdot \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \cdot \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix} \qquad c_{1,2} \in \mathbb{R}$$

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b) $x = c_1(t) \cdot (\cos t - \sin t) + c_2(t) \cdot (\cos t + \sin t)$

$y = c_1(t) \cdot \cos t + c_2(t) \cdot \sin t$

$$\begin{pmatrix} \cos t - \sin t & \cos t + \sin t \\ \cos t & \sin t \end{pmatrix} \cdot \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{vmatrix} \cos t - \sin t & \cos t + \sin t \\ \cos t & \sin t \end{vmatrix} = \cos t \cdot \sin t - \sin^2 t - \cos^2 t - \cos t \cdot \sin t = -1 \neq 0$$

$$c_1'(t) = \frac{1}{-1} \cdot \begin{vmatrix} 0 & \cos t + \sin t \\ 1 & \sin t \end{vmatrix} = \cos t + \sin t$$

$$c_2'(t) = \frac{1}{-1} \cdot \begin{vmatrix} \cos t - \sin t & 0 \\ \cos t & 1 \end{vmatrix} = -\cos t + \sin t$$

$$c_1(t) = \int (\cos t + \sin t) \cdot dt = \sin t - \cos t$$

$$c_2(t) = \int (-\cos t + \sin t) \cdot dt = -\sin t - \cos t$$

$$x = (\sin t - \cos t) \cdot (\cos t - \sin t) + (-\sin t - \cos t) \cdot (\cos t + \sin t) =$$

$$= \sin t \cdot \cos t - \sin^2 t - \cos^2 t + \sin t \cdot \cos t - \sin t \cdot \cos t - \sin^2 t - \cos^2 t - \cos t \cdot \sin t =$$

$$= -2$$

$$y = (\sin t - \cos t) \cdot \cos t + (-\sin t - \cos t) \cdot \sin t =$$

$$= \sin t \cdot \cos t - \cos^2 t - \sin^2 t - \cos t \cdot \sin t =$$

$$= -1$$

c) $$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \cdot \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \cdot \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$x = (c_1 + c_2) \cdot \cos t + (-c_1 + c_2) \cdot \sin t - 2$$
$$y = c_1 \cdot \cos t + c_2 \cdot \sin t - 1$$

$c_{1,2} \in \mathbb{R}$

$$③\quad \begin{aligned} x' &= 2x - y + 8e^{-2t} \\ y' &= x + 4y + 2e^{-2t} \end{aligned}$$

a) $\begin{vmatrix} 2-r & -1 \\ 1 & 4-r \end{vmatrix} = 0$

$$(2-r)(4-r)+1=0$$
$$r^2-6r+9=0$$
$$(r-3)^2=0 \quad \Rightarrow \quad r_1=r_2=3$$

$r_1=r_2=3$: $\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad v_1+v_2=0$, $\quad \bar{v}=(1,-1)$

1. řešení: $\begin{pmatrix} x \\ y \end{pmatrix} = e^{3t} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$ $\quad w_1+w_2=-1$, $\quad \bar{w}=(1,-2)$ $\quad$ zobecněný vlastní v.

2. řešení: $\begin{pmatrix} x \\ y \end{pmatrix} = e^{3t} \cdot \left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} (t+1) \cdot e^{3t} \\ (-t-2) \cdot e^{3t} \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \cdot \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix} + c_2 \cdot \begin{pmatrix} (t+1) \cdot e^{3t} \\ (-t-2) \cdot e^{3t} \end{pmatrix}$$

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b) $x = c_1(t) \cdot e^{3t} + c_2(t) \cdot (t+1) \cdot e^{3t}$

$\ y = -c_1(t) \cdot e^{3t} + c_2(t) \cdot (-t-2) \cdot e^{3t}$

$$\begin{pmatrix} e^{3t} & (t+1) \cdot e^{3t} \\ -e^{3t} & (-t-2)e^{3t} \end{pmatrix} \cdot \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 8e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

$$\begin{vmatrix} e^{3t} & (t+1) \cdot e^{3t} \\ -e^{3t} & (-t-2)e^{3t} \end{vmatrix} = e^{3t} \cdot (-t-2) \cdot e^{3t} + e^{3t} \cdot (t+1) \cdot e^{3t} = -e^{6t} \neq 0$$

$$c_1'(t) = \frac{1}{-e^{6t}} \cdot \begin{vmatrix} 8e^{-2t} & (t+1)\cdot e^{3t} \\ 2e^{-2t} & (-t-2)\cdot e^{3t} \end{vmatrix} = -e^{-6t} \cdot \left((-8t-16)\cdot e^{t} - (2t+2)\cdot e^{t} \right) =$$

$$= -e^{-6t} \cdot (-10t-18)\cdot e^{t} = (10t+18)\cdot e^{-5t}$$

$$c_2'(t) = \frac{1}{-e^{6t}} \cdot \begin{vmatrix} e^{3t} & 8e^{-2t} \\ -e^{3t} & 2e^{-2t} \end{vmatrix} = -e^{-6t} \cdot \left(2e^{t} + 8e^{t} \right) = -10e^{-5t}$$

$$c_1(t) = \int (10t+18)\cdot e^{-5t}\cdot dt = \begin{vmatrix} u = 10t+18 & v' = e^{-5t} \\ u' = 10 & v = \frac{e^{-5t}}{-5} \end{vmatrix} =$$

$$= (10t+18)\cdot \frac{e^{-5t}}{-5} - \int 10 \cdot \frac{e^{-5t}}{-5} \cdot dt = \left(-2t - \frac{18}{5}\right) e^{-5t} + \int 2e^{-5t} \cdot dt =$$

$$= \left(-2t - \frac{18}{5}\right) e^{-5t} - \frac{2}{5} e^{-5t} = (-2t-4)e^{-5t}$$

$$c_2(t) = \int -10e^{-5t} \cdot dt = \frac{-10e^{-5t}}{-5} = 2e^{-5t}$$

$$x = (-2t-4)\cdot e^{-5t} \cdot e^{3t} + 2e^{-5t} \cdot (t+1)\cdot e^{3t} = -2e^{-2t}$$

$$y = -(-2t-4)e^{-5t} \cdot e^{3t} + 2e^{-5t} \cdot (-t-2)\cdot e^{3t} = 0$$

c) $$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \cdot \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix} + c_2 \cdot \begin{pmatrix} (t+1)e^{3t} \\ (-t-2)e^{3t} \end{pmatrix} + \begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix} \qquad c_{1,2} \in \mathbb{R}$$

$$x = (c_1 + c_2)\cdot e^{3t} + c_2 \cdot t \cdot e^{3t} - 2e^{-2t}$$

$$y = (-c_1 - 2c_2)\cdot e^{3t} - 2c_2 \cdot t \cdot e^{3t} \qquad c_{1,2} \in \mathbb{R}$$