Soustavy obyčejných diferenciálních rovnic

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Řešené příklady

1.
$$\begin{aligned} x' &= 2x - y \\ y' &= 4x - 3y \end{aligned} \qquad \begin{aligned} x(0) &= 1 \\ y(0) &= -5 \end{aligned}$$

2.
$$\begin{aligned} x' &= -x - y \\ y' &= 9x + 5y \end{aligned} \qquad \begin{aligned} x(0) &= 2 \\ y(0) &= -1 \end{aligned}$$

3.
$$\begin{aligned} x' &= 4x - y \\ y' &= 18x - 2y \end{aligned} \qquad \begin{aligned} x(0) &= 3 \\ y(0) &= 6 \end{aligned}$$

4.
$$\begin{aligned} x' &= -2x + y \\ y' &= -2x + y + 1 \end{aligned} \qquad \begin{aligned} x(0) &= 0 \\ y(0) &= 0 \end{aligned}$$

5.
$$\begin{aligned} x' &= x - 5y \\ y' &= x - y + 6e^{2t} \end{aligned} \qquad \begin{aligned} x(0) &= 0 \\ y(0) &= 1 \end{aligned}$$

6.
$$\begin{aligned} x' &= 3x - y + e^{-t} \\ y' &= 4x - y \end{aligned} \qquad \begin{aligned} x(0) &= 3 \\ y(0) &= 2 \end{aligned}$$

① $x' = 2x - y \qquad x(0) = 1$

$y' = 4x - 3y \qquad y(0) = -5$

$y = -x' + 2x \qquad (*)$

$y' = -x'' + 2x'$

$-x'' + 2x' = 4x - 3(-x' + 2x)$

$x'' + x' - 2x = 0$

$r^2 + r - 2 = 0$

$(r-1)(r+2) = 0 \quad \Rightarrow r_1 = 1,\ r_2 = -2$

$x = c_1 e^{t} + c_2 e^{-2t} \qquad c_{1,2} \in \mathbb{R}$

$x' = c_1 e^{t} - 2c_2 e^{-2t}$

$(*)$: $y = -x' + 2x = -(c_1 e^{t} - 2c_2 e^{-2t}) + 2(c_1 e^{t} + c_2 e^{-2t}) = c_1 e^{t} + 4c_2 e^{-2t}$

$x = c_1 e^{t} + c_2 e^{-2t}$

$y = c_1 e^{t} + 4c_2 e^{-2t} \qquad c_{1,2} \in \mathbb{R}$

$x(0) = 1 \ : \ 1 = c_1 e^0 + c_2 e^0$

$y(0) = -5 \ : \ -5 = c_1 e^0 + 4c_2 e^0$

$1 = c_1 + c_2$

$-5 = c_1 + 4c_2 \quad \Rightarrow \quad c_1 = 3,\ c_2 = -2$

$x = 3e^{t} - 2e^{-2t}$

$y = 3e^{t} - 8e^{-2t}$

② $x' = -x - y \qquad x(0) = 2$

$y' = 9x + 5y \qquad y(0) = -1$

$y = -x' - x \quad (*)$

$y' = -x'' - x'$

$-x'' - x' = 9x + 5(-x' - x)$

$x'' - 4x' + 4x = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 2$

$x = c_1 e^{2t} + c_2 t e^{2t} \qquad c_{1,2} \in \mathbb{R}$

$x' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}$

$(*)$: $y = -x' - x = -2c_1 e^{2t} - c_2 e^{2t} - 2c_2 t e^{2t} - c_1 e^{2t} - c_2 t e^{2t} =$

$= (-3c_1 - c_2) e^{2t} - 3c_2 t e^{2t}$

$x = c_1 e^{2t} + c_2 t e^{2t}$

$y = (-3c_1 - c_2) e^{2t} - 3c_2 t e^{2t} \qquad c_{1,2} \in \mathbb{R}$

$x(0) = 2 \quad : \quad 2 = c_1$

$y(0) = -1 \quad : \quad -1 = -3c_1 - c_2 \qquad \Rightarrow \qquad c_1 = 2, \quad c_2 = -5$

$x = 2e^{2t} - 5te^{2t}$

$y = -e^{2t} + 15te^{2t}$

3

$$x' = 4x - y \qquad x(0) = 3$$
$$y' = 18x - 2y \qquad y(0) = 6$$

$$y = -x' + 4x \quad (*)$$
$$y' = -x'' + 4x'$$
$$-x'' + 4x' = 18x - 2(-x' + 4x)$$
$$x'' - 2x' + 10x = 0$$
$$r^2 - 2r + 10 = 0 \qquad r_{1,2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$$x = c_1 e^t \cos 3t + c_2 e^t \sin 3t \qquad c_{1,2} \in \mathbb{R}$$
$$x' = c_1 e^t \cos 3t - 3c_1 e^t \sin 3t + c_2 e^t \sin 3t + 3c_2 e^t \cos 3t$$

$$(*) \quad y = -x' + 4x =$$
$$= -c_1 e^t \cos 3t + 3c_1 e^t \sin 3t - c_2 e^t \sin 3t - 3c_2 e^t \cos 3t +$$
$$+ 4(c_1 e^t \cos 3t + c_2 e^t \sin 3t)$$
$$y = (3c_1 - 3c_2) e^t \cos 3t + (3c_1 + 3c_2) e^t \sin 3t$$

$$x = c_1 e^t \cos 3t + c_2 e^t \sin 3t$$
$$y = (3c_1 - 3c_2) e^t \cos 3t + (3c_1 + 3c_2) e^t \sin 3t \qquad c_{1,2} \in \mathbb{R}$$

$$x(0) = 3 \;:\; 3 = c_1$$
$$y(0) = 6 \;:\; 6 = 3c_1 - 3c_2 \quad \Rightarrow \quad c_1 = 3, \; c_2 = 1$$

$$x = 3e^t \cos 3t + e^t \sin 3t$$
$$y = 6e^t \cos 3t + 12 e^t \sin 3t$$

4

$$x' = -2x + y \qquad x(0) = 0$$
$$y' = -2x + y + 1 \qquad y(0) = 0$$

$$y = x' + 2x \quad (*)$$
$$y' = x'' + 2x'$$
$$x'' + 2x' = -2x + x' + 2x + 1$$
$$x'' + x' = 1 \quad (0)$$

a) $r^2 + r = 0$

$r(r+1) = 0 \Rightarrow r_1 = 0,\ r_2 = -1$

$x = c_1 e^{0t} + c_2 e^{-t} = c_1 + c_2 e^{-t}$

b) $f(t) = 1 \Rightarrow y = At \qquad (r_1 = 0!)$

$y' = A$

$y'' = 0$

$(0):\ 0 + A = 1 \Rightarrow A = 1 \Rightarrow y = t$

c) $x = c_1 + c_2 e^{-t} + t \qquad c_{1,2} \in \mathbb{R}$

$x' = -c_2 e^{-t} + 1$

$(*):\ y = x' + 2x = -c_2 e^{-t} + 1 + 2(c_1 + c_2 e^{-t} + t) = 2c_1 + c_2 e^{-t} + 2t + 1$

$$x = c_1 + c_2 e^{-t} + t$$
$$y = 2c_1 + c_2 e^{-t} + 2t + 1 \qquad c_{1,2} \in \mathbb{R}$$

$x(0) = 0 \ :\ 0 = c_1 + c_2$

$y(0) = 0 \ :\ 0 = 2c_1 + c_2 + 1$

$\Rightarrow c_1 = -1,\ c_2 = 1$

$$x = e^{-t} + t - 1$$
$$y = e^{-t} + 2t - 1$$

5.

$$x' = x - 5y \qquad x(0) = 0$$
$$y' = x - y + 6e^{2t} \qquad y(0) = 1$$

$$\rightarrow x = y' + y - 6e^{2t} \quad (*)$$
$$x' = y'' + y' - 12e^{2t}$$

$$\rightarrow y'' + y - 12e^{2t} = y' + y - 6e^{2t} - 5y$$
$$y'' + 4y = 6e^{2t} \quad (o)$$

a) $r^2 + 4r = 0$

$r_{1,2} = \pm 2i$

$y = c_1 \cos 2t + c_2 \sin 2t$

b) $f(t) = 6e^{2t} \Rightarrow y = Ae^{2t}$

$y' = 2Ae^{2t}$

$y'' = 4Ae^{2t}$

$(o):\ 4Ae^{2t} + 4Ae^{2t} = 6e^{2t}$

$8Ae^{2t} = 6e^{2t} \Rightarrow A = \frac{3}{4}$

$y = \frac{3}{4} \cdot e^{2t}$

c) $y = c_1 \cdot \cos 2t + c_2 \cdot \sin 2t + \frac{3}{4} \cdot e^{2t} \quad c_{1,2} \in \mathbb{R}$

$y' = -2c_1 \cdot \sin 2t + 2c_2 \cdot \cos 2t + \frac{3}{2} \cdot e^{2t}$

$(*):\ x = -2c_1 \cdot \sin 2t + 2c_2 \cdot \cos 2t + \frac{3}{2} \cdot e^{2t} + c_1 \cdot \cos 2t + c_2 \cdot \sin 2t + \frac{3}{4} \cdot e^{2t} - 6e^{2t}$

$x = (c_1 + 2c_2) \cdot \cos 2t + (-2c_1 + c_2) \cdot \sin 2t - \frac{15}{4} \cdot e^{2t}$

$$x = (c_1 + 2c_2) \cdot \cos 2t + (-2c_1 + c_2) \cdot \sin 2t - \frac{15}{4} \cdot e^{2t}$$
$$y = c_1 \cdot \cos 2t + c_2 \cdot \sin 2t + \frac{3}{4} \cdot e^{2t}$$

$x(0) = 0 \ :\ 0 = c_1 + 2c_2 - \frac{15}{4}$

$y(0) = 1 \ :\ 1 = c_1 + \frac{3}{4}$

$\Rightarrow c_1 = \frac{1}{4}, \quad c_2 = \frac{7}{4}$

$$x = \frac{15}{4} \cdot \cos 2t + \frac{5}{4} \cdot \sin 2t - \frac{15}{4} \cdot e^{2t}$$
$$y = \frac{1}{4} \cdot \cos 2t + \frac{7}{4} \cdot \sin 2t + \frac{3}{4} \cdot e^{2t}$$

6

$$x' = 3x - y + e^{-t} \qquad x(0) = 3$$
$$y' = 4x - y \qquad y(0) = 2$$

$$y = -x' + 3x + e^{-t} \quad (*)$$
$$y' = -x'' + 3x' - e^{-t}$$
$$-x'' + 3x' - e^{-t} = 4x - (-x' + 3x + e^{-t})$$

$$x'' - 2x' + x = 0$$
$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 1$$

$$x = c_1 e^t + c_2 t e^t \qquad c_{1,2} \in \mathbb{R}$$
$$x' = c_1 e^t + c_2 e^t + c_2 t e^t$$

$$(*): \quad y = -x' + 3x + e^{-t} = -c_1 e^t - c_2 e^t - c_2 t e^t + 3c_1 e^t + 3c_2 t e^t + e^{-t}$$
$$y = (2c_1 - c_2)e^t + 2c_2 t e^t + e^{-t}$$

$$x = c_1 e^t + c_2 t e^t$$
$$y = (2c_1 - c_2)e^t + 2c_2 t e^t + e^{-t} \qquad c_{1,2} \in \mathbb{R}$$

$$x(0) = 3 : \quad 3 = c_1$$
$$y(0) = 2 : \quad 2 = 2c_1 - c_2 + 1 \quad \Rightarrow \quad c_1 = 3, \; c_2 = 5$$

$$x = 3e^t + 5te^t$$
$$y = e^t + 10te^t + e^{-t}$$