

$$\begin{aligned}
 \textcircled{1} \quad \iint_{\Omega} (9 - x^2 - y^2) \cdot dx dy &= \int_0^1 \left(\int_0^2 (9 - x^2 - y^2) dy \right) dx = \\
 \Omega: \langle 0, 1 \rangle \times \langle 0, 2 \rangle &= \int_0^1 dx \int_0^2 (9 - x^2 - y^2) dy = \int_0^1 \left[9y - x^2y - \frac{y^3}{3} \right]_0^2 dx = \\
 &= \int_0^1 \left(18 - 2x^2 - \frac{8}{3} \right) dx = \int_0^1 \left(\frac{46}{3} - 2x^2 \right) dx = \\
 &= \left[\frac{46x}{3} - \frac{2x^3}{3} \right]_0^1 = \frac{46}{3} - \frac{2}{3} = \underline{\underline{\frac{44}{3}}}
 \end{aligned}$$

Co jsme spočítali?

Objem "hranolu" mezi Ω v rovině $z=0$ a plochou $z=9-x^2-y^2$

jak vypadá plocha $z=9-x^2-y^2$?

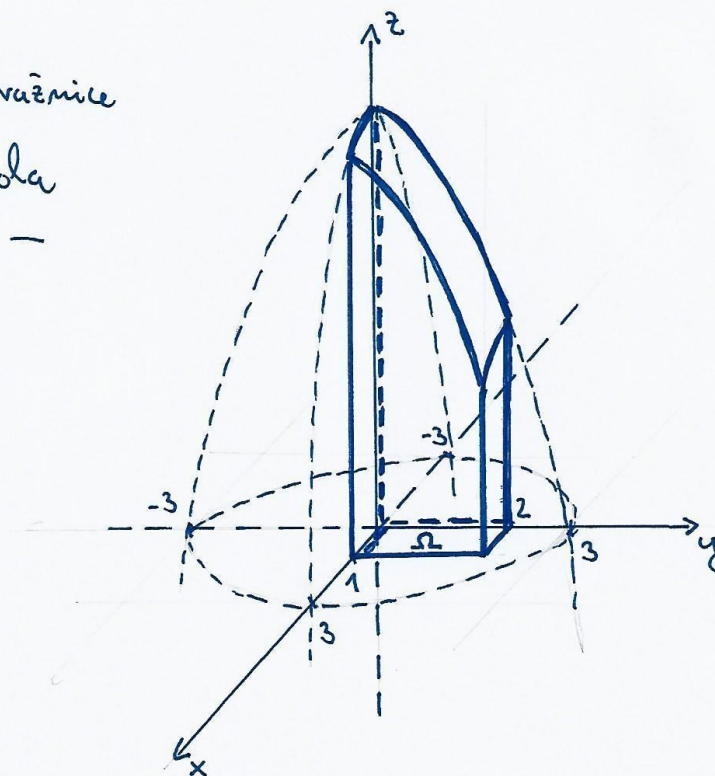
Určíme řezy v souřadných rovinách:

$$\begin{aligned}
 z=0: \quad 0 &= 9 - x^2 - y^2 \\
 x^2 + y^2 &= 9 \quad \text{okružnice}
 \end{aligned}$$

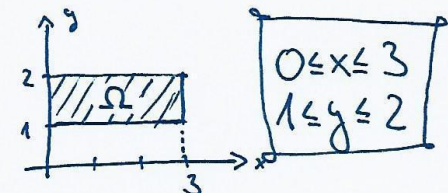
$$x=0: \quad z=9-y^2 \quad \text{parabola}$$

$$y=0: \quad z=9-x^2 \quad \text{---}$$

rotací parabolická
plocha



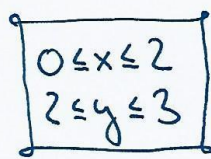
$$\begin{aligned}
 \textcircled{2} \quad \iint_{\Omega} \frac{x^2}{y^2} \cdot dx dy &= \int_0^3 dx \int_1^2 \frac{x^2}{y^2} \cdot dy = \int_0^3 \left[-\frac{x^2}{y} \right]_1^2 \cdot dx = \\
 \Omega: \langle 0,3 \rangle \times \langle 1,2 \rangle &= \int_0^3 \left(-\frac{x^2}{2} + \frac{x^2}{1} \right) dx = \int_0^3 \frac{x^2}{2} \cdot dx = \\
 &= \left[\frac{x^3}{6} \right]_0^3 = \frac{27}{6} = \underline{\underline{\frac{9}{2}}}
 \end{aligned}$$



podmínka $y \neq 0$ ✓

$$\begin{aligned}
 \text{nebo: } \iint_{\Omega} \frac{x^2}{y^2} dx dy &= \int_0^3 x^2 \cdot dx \cdot \int_1^2 \frac{1}{y^2} \cdot dy = \left[\frac{x^3}{3} \right]_0^3 \cdot \left[-\frac{1}{y} \right]_1^2 = \\
 &= \frac{27}{3} \cdot \left(-\frac{1}{2} + 1 \right) = \underline{\underline{\frac{9}{2}}}
 \end{aligned}$$

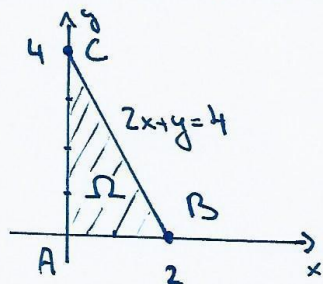
$$\begin{aligned}
 \textcircled{3} \quad \iint_{\Omega} \frac{y^2+4}{x^2+4} \cdot dx dy &= \int_0^2 \frac{1}{x^2+4} \cdot dx \cdot \int_2^3 (y^2+4) \cdot dy = \\
 \Omega: \langle 0,2 \rangle \times \langle 2,3 \rangle &= \left[\frac{1}{2} \cdot \operatorname{arctg} \frac{x}{2} \right]_0^2 \cdot \left[\frac{y^3}{3} + 4y \right]_2^3 = \\
 &= \left(\frac{1}{2} \operatorname{arctg} 1 - \frac{1}{2} \operatorname{arctg} 0 \right) \cdot \left(9 + 12 - \frac{8}{3} - 8 \right) = \\
 &= \frac{\pi}{8} \cdot \frac{31}{3} = \underline{\underline{\frac{31\pi}{24}}}
 \end{aligned}$$



$x^2+4 \neq 0$ ✓

$$\textcircled{4} \iint_{\Omega} 2xy \cdot dx dy$$

$\Omega: \triangle ABC, A[0;0], B[2;0], C[0;4]$



$$\begin{array}{|l} 0 \leq x \leq 2 \\ 0 \leq y \leq 4-2x \end{array}$$

nebo

$$\begin{array}{|l} 0 \leq y \leq 4 \\ 0 \leq x \leq 2 - \frac{y}{2} \end{array}$$

rovnice přímky BC:

parametricky $X = B + (C-B) \cdot t$

$$x = 2 - 2t \quad / \cdot 2$$

$$y = 0 + 4t$$

$$2x + y = 4$$

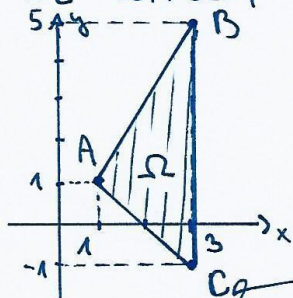
$$\iint_{\Omega} 2xy \cdot dx dy = \int_0^2 dx \int_0^{4-2x} 2xy \cdot dy = \int_0^2 \left[xy^2 \right]_0^{4-2x} dx = \int_0^2 x(4-2x)^2 dx =$$

$$= \int_0^2 (16x - 16x^2 + 4x^3) dx = \left[8x^2 - \frac{16x^3}{3} + x^4 \right]_0^2 =$$

$$= 32 - \frac{128}{3} + 16 = \underline{\underline{\frac{16}{3}}}$$

$$\textcircled{5} \iint_{\Omega} x^2 \cdot dx dy$$

$\Omega: \triangle ABC, A[1;1], B[3;5], C[3;-1]$



$$AB: \begin{aligned} x &= 1+2t \\ y &= 1+4t \end{aligned} \quad | \cdot (-2) \\ \hline -2x+y=-1$$

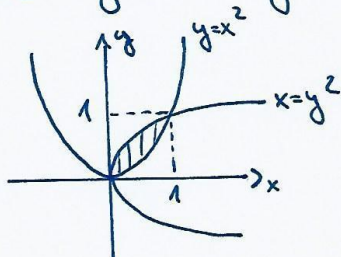
$$AC: \begin{aligned} x &= 1+2t \\ y &= 1-2t \end{aligned} \\ \hline x+y=2$$

$$\begin{aligned} 1 \leq x \leq 3 \\ 2-x \leq y \leq 2x-1 \end{aligned} \quad \text{nebo} \quad \begin{pmatrix} -1 \leq y \leq 1 & 1 \leq y \leq 5 \\ 2-y \leq x \leq 3 & \frac{y+1}{2} \leq x \leq 3 \end{pmatrix}$$

$$\begin{aligned} \iint_{\Omega} x^2 \cdot dx dy &= \int_1^3 dx \int_{2-x}^{2x-1} x^2 \cdot dy = \int_1^3 \left[x^2 y \right]_{2-x}^{2x-1} dx = \\ &= \int_1^3 (x^2(2x-1) - x^2(2-x)) dx = \int_1^3 (3x^3 - 3x^2) dx = \\ &= \left[\frac{3x^4}{4} - x^3 \right]_1^3 = \frac{243}{4} - 27 - \frac{3}{4} + 1 = \underline{\underline{34}} \end{aligned}$$

$$⑥ \iint_{\Omega} (x^2 + y^2) \cdot dx dy$$

$$\Omega: y = x^2, x = y^2$$



$$\boxed{\begin{matrix} 0 \leq x \leq 1 \\ x^2 \leq y \leq \sqrt{x} \end{matrix}} \quad \text{nebo} \quad \left(\begin{matrix} 0 \leq y \leq 1 \\ y^2 \leq x \leq \sqrt{y} \end{matrix} \right)$$

průsečík: $y = x^2, x = y^2$

$$x = (x^2)^2$$

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0$$

$$x = 0 \quad x = 1$$

$$\iint_{\Omega} (x^2 + y^2) dx dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x^2 + y^2) dy =$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^{\sqrt{x}} dx =$$

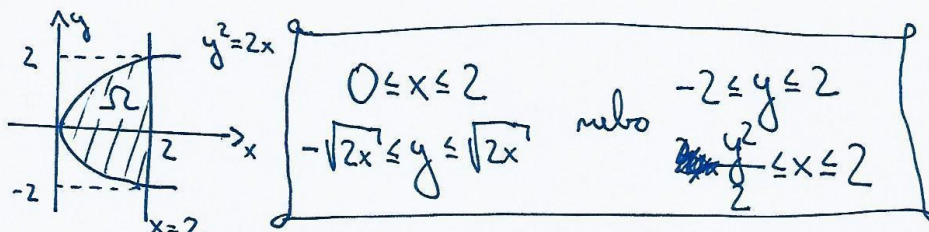
$$= \int_0^1 \left(x^2 \sqrt{x} + \frac{(\sqrt{x})^3}{3} - x^2 \cdot x^2 - \frac{(x^2)^3}{3} \right) dx =$$

$$= \int_0^1 \left(x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - x^4 - \frac{x^6}{3} \right) dx = \left[\frac{2x^{\frac{7}{2}}}{7} + \frac{2x^{\frac{5}{2}}}{15} - \frac{x^5}{5} - \frac{x^7}{21} \right]_0^1 =$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} = \underline{\underline{\frac{6}{35}}}$$

$$\textcircled{7} \iint_{\Omega} y \cdot dx dy = \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} y \cdot dy = \int_0^2 \left[\frac{y^2}{2} \right]_{-\sqrt{2x}}^{\sqrt{2x}} dx = \int_0^2 (x - x) dx = \underline{\underline{0}}$$

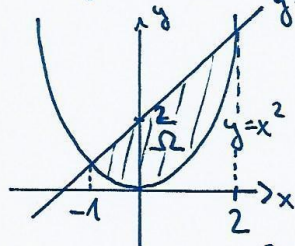
$$\Omega: y^2 = 2x, x = 2$$



$$\begin{aligned} \text{nebo } \iint_{\Omega} y \cdot dx dy &= \int_{-2}^2 dy \int_{\frac{y^2}{2}}^2 y dx = \int_{-2}^2 [yx]_{\frac{y^2}{2}}^2 dy = \int_{-2}^2 (2y - \frac{y^3}{2}) dy = \\ &= \left[y^2 - \frac{y^4}{8} \right]_{-2}^2 = 4 - 2 - 4 + 2 = \underline{\underline{0}} \end{aligned}$$

$$\textcircled{8} \iint_{\Omega} x \cdot dx dy = \int_{-1}^2 dx \int_{x^2}^{x+2} x \cdot dy = \int_{-1}^2 [xy]_{x^2}^{x+2} dx =$$

$$\Omega: y = x^2, y = x+2 \quad = \int_{-1}^2 (x^2 + 2x - x^3) \cdot dx = \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4} \right]_{-1}^2 =$$

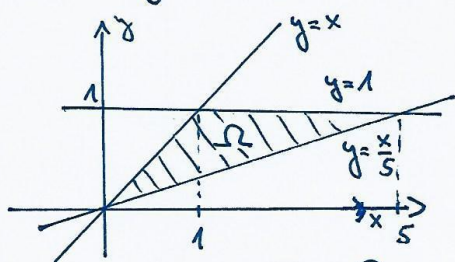


$$= \frac{8}{3} + 4 - 4 - \left(-\frac{1}{3} + 1 - \frac{1}{4} \right) = \underline{\underline{\frac{9}{4}}}$$

$$\begin{aligned} x^2 &= x+2 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x &= -1 \quad x = 2 \end{aligned}$$

$$\textcircled{9} \iint_{\Omega} \sqrt{xy - y^2} \, dx \, dy = \int_0^1 dy \int_y^{5y} \sqrt{xy - y^2} \, dx =$$

$$\Omega: y=x, y=\frac{x}{5}, y=1$$



$$\boxed{\begin{array}{l} 0 \leq y \leq 1 \\ y \leq x \leq 5y \end{array}}$$

$$= \left| \begin{array}{l} xy - y^2 = k \\ y \cdot dx = dk \\ dx = \frac{1}{y} dk \\ x = 5y \Rightarrow k = 4y^2 \\ x = y \Rightarrow k = 0 \end{array} \right| =$$

$$= \int_0^1 dy \int_0^{4y^2} \sqrt{k} \cdot \frac{1}{y} \cdot dk = \int_0^1 \left[\frac{1}{y} \cdot \frac{2k^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{4y^2} dy =$$

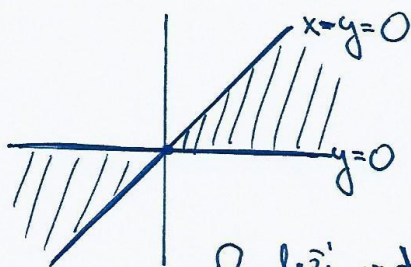
$$= \int_0^1 \frac{1}{y} \cdot \frac{2(4y^2)^{\frac{3}{2}}}{3} \cdot dy = \frac{16}{3} \cdot \int_0^1 y^2 \cdot dy = \frac{16}{3} \cdot \left[\frac{y^3}{3} \right]_0^1 =$$

$$= \frac{16}{3} \cdot \frac{1}{3} = \frac{16}{9}$$

podmínky:

$$xy - y^2 \geq 0$$

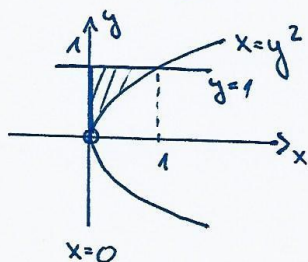
$$y(x - y) \geq 0$$



Ω leží v def. oboru

$$\textcircled{10} \iint_{\Omega} e^{\frac{x}{y}} \cdot dx dy = \int_0^1 dy \int_0^{y^2} e^{\frac{x}{y}} dx = \left| \begin{array}{l} \frac{x}{y} = k \\ \frac{1}{y} dx = dk \\ dx = y \cdot dk \\ x = y^2 \Rightarrow k = y \\ x = 0 \Rightarrow k = 0 \end{array} \right| =$$

$$\Omega: x=y^2, y=1, x=0$$



Pozn.: máme na výběr
 $\int e^x \cdot dx \quad ;$

nebo $\int e^{\frac{1}{y}} dy \quad ;$

proto

$$\boxed{\begin{array}{l} 0 \leq y \leq 1 \\ 0 \leq x \leq y^2 \end{array}}$$

$$\begin{aligned} &= \int_0^1 dy \int_0^y e^k \cdot y \cdot dk = \int_0^1 [y \cdot e^k]_0^y dy = \\ &= \int_0^1 (y \cdot e^y - y) dy = \left| \begin{array}{ll} u=y & v'=e^y \\ u'=1 & v=e^y \end{array} \right| = \\ &= [y \cdot e^y]_0^1 - \int_0^1 y \cdot dy - \left[\frac{y^2}{2} \right]_0^1 = \\ &= e - [e^y]_0^1 - \frac{1}{2} = e - (e-1) - \frac{1}{2} = \underline{\underline{\frac{1}{2}}} \end{aligned}$$