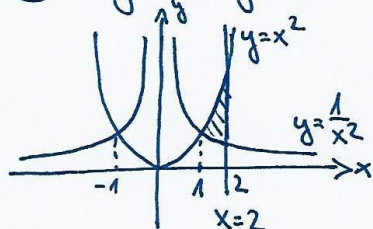


Obrab plochy 2D $S = \iint_{\Omega} dx dy$

① $y = x^2, y = \frac{1}{x^2}, x = 2$



$$x^2 = \frac{1}{x^2} \Rightarrow x^4 = 1 \Rightarrow x = \pm 1$$

$$\boxed{1 \leq x \leq 2}$$

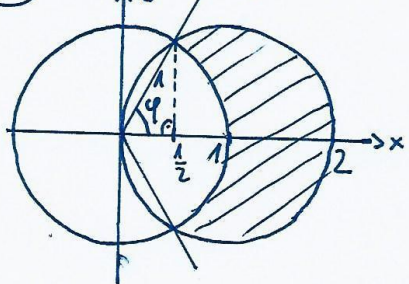
$$\boxed{\frac{1}{x^2} \leq y \leq x^2}$$

$$S = \iint_{\Omega} dx dy = \int_1^2 dx \int_{\frac{1}{x^2}}^{x^2} dy =$$

$$= \int_1^2 [y]_{\frac{1}{x^2}}^{x^2} \cdot dx = \int_1^2 \left(x^2 - \frac{1}{x^2} \right) \cdot dx =$$

$$= \left[\frac{x^3}{3} + \frac{1}{x} \right]_1^2 = \frac{8}{3} + \frac{1}{2} - \frac{1}{3} - 1 = \frac{11}{6}$$

② $1 \leq x^2 + y^2 \leq 2x$



$$x = \rho \cdot \cos \varphi$$

$$y = \rho \cdot \sin \varphi$$

$$\rho = \rho$$

$$x^2 + y^2 = 2x$$

$$\rho^2 = 2\rho \cos \varphi$$

$$\rho = 2 \cos \varphi$$

$$\varphi: \cos \varphi = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\varphi = \frac{\pi}{3}$$

$$\boxed{-\frac{\pi}{3} \leq \varphi \leq \frac{\pi}{3}}$$

$$\boxed{1 \leq \rho \leq 2 \cos \varphi}$$

resp. $0 \leq \varphi \leq \frac{\pi}{3}$ a $2x$

$$S = \iint_{\Omega} dx dy = 2 \cdot \int_0^{\frac{\pi}{3}} d\varphi \int_1^{2 \cos \varphi} \rho \cdot d\rho = 2 \cdot \int_0^{\frac{\pi}{3}} \left[\frac{\rho^2}{2} \right]_1^{2 \cos \varphi} \cdot d\varphi =$$

$$= 2 \cdot \int_0^{\frac{\pi}{3}} \left(2 \cos^2 \varphi - \frac{1}{2} \right) d\varphi = 2 \cdot \int_0^{\frac{\pi}{3}} \left(2 \cdot \frac{1 + \cos 2\varphi}{2} - \frac{1}{2} \right) \cdot d\varphi =$$

$$= 2 \cdot \int_0^{\frac{\pi}{3}} \left(1 + \cos 2\varphi - \frac{1}{2} \right) \cdot d\varphi = 2 \cdot \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} + \cos 2\varphi \right) \cdot d\varphi =$$

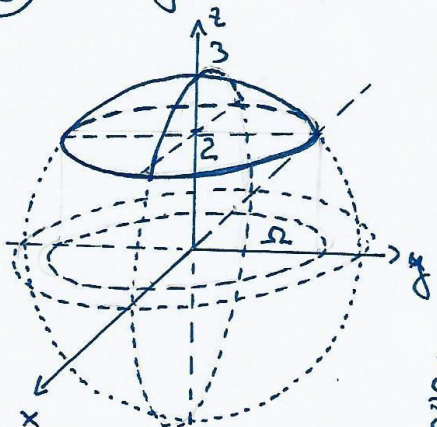
$$= 2 \cdot \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{2} \right]_0^{\frac{\pi}{3}} = 2 \cdot \left(\frac{\pi}{6} + \frac{\sin \frac{2\pi}{3}}{2} - 0 \right) = 2 \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right) =$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

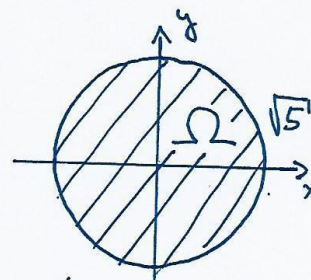
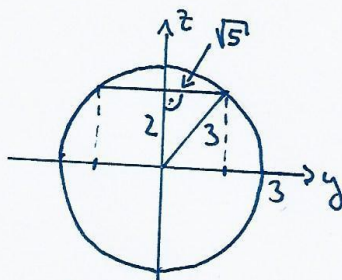
Obsah plochy Σ 3D : $z = f(x, y)$ nad Ω

$$S = \iint_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

③ $x^2 + y^2 + z^2 = 9, z \geq 2$



$$\begin{aligned} x &= \rho \cdot \cos \varphi \\ y &= \rho \cdot \sin \varphi \\ \rho &= \rho \end{aligned} \quad \begin{cases} 0 \leq \rho \leq \sqrt{5} \\ 0 \leq \varphi \leq 2\pi \end{cases}$$



$$z = \sqrt{9 - x^2 - y^2} = (9 - x^2 - y^2)^{\frac{1}{2}}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(9 - x^2 - y^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{9 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(9 - x^2 - y^2)^{-\frac{1}{2}} \cdot (-2y) = \frac{-y}{\sqrt{9 - x^2 - y^2}}$$

$$S = \iint_{\Omega} \sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{9 - x^2 - y^2}}\right)^2} \cdot dx dy =$$

$$= \iint_{\Omega} \sqrt{1 + \frac{x^2}{9 - x^2 - y^2} + \frac{y^2}{9 - x^2 - y^2}} \cdot dx dy = \iint_{\Omega} \sqrt{\frac{9 - x^2 - y^2 + x^2 + y^2}{9 - x^2 - y^2}} \cdot dx dy =$$

$$= \iint_{\Omega} \frac{3}{\sqrt{9 - x^2 - y^2}} \cdot dx dy = \iint_{\Omega^*} \frac{3}{\sqrt{9 - \rho^2 \cos^2 \varphi - \rho^2 \sin^2 \varphi}} \cdot \rho \cdot d\rho d\varphi =$$

$$= 3 \iint_{\Omega} \frac{1}{\sqrt{9 - \rho^2}} \cdot \rho \cdot d\rho d\varphi = 3 \cdot \int_0^{\sqrt{5}} \frac{1}{\sqrt{9 - \rho^2}} \cdot \rho \cdot d\rho \cdot \int_0^{2\pi} d\varphi = \left. \begin{array}{l} 9 - \rho^2 = k \\ -2\rho d\rho = dk \\ \rho \cdot d\rho = -\frac{1}{2} dk \\ \rho = \sqrt{5} \Rightarrow k = 4 \\ \rho = 0 \Rightarrow k = 9 \end{array} \right| =$$

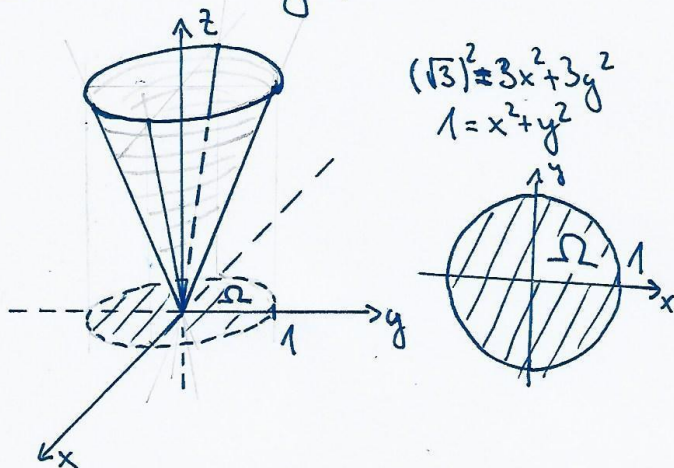
$$= 3 \cdot \int_4^9 \frac{1}{\sqrt{k}} \cdot \left(-\frac{1}{2}\right) \cdot dk \cdot [\varphi]_0^{2\pi} =$$

$$= 3 \cdot 2\pi \cdot \left(-\frac{1}{2}\right) \cdot \int_9^4 t^{-\frac{1}{2}} dt = -3\pi \left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right]_9^4 =$$

$$= -3\pi \cdot 2 \cdot (2 - 3) = \underline{\underline{6\pi}}$$

(4) $z^2 = 3x^2 + 3y^2, z \geq 0, z \leq \sqrt{3}$

$\left(\begin{array}{l} x=0: z^2 = 3y^2 \\ z = \pm \sqrt{3} \cdot y \end{array} \right)$



$$z = \sqrt{3x^2 + 3y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2} \cdot (3x^2 + 3y^2)^{-\frac{1}{2}} \cdot 6x =$$

$$= \frac{3x}{\sqrt{3x^2 + 3y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{3y}{\sqrt{3x^2 + 3y^2}}$$

$$S = \iint_{\Omega} \sqrt{1 + \frac{9x^2}{3x^2 + 3y^2} + \frac{9y^2}{3x^2 + 3y^2}} \cdot dx dy =$$

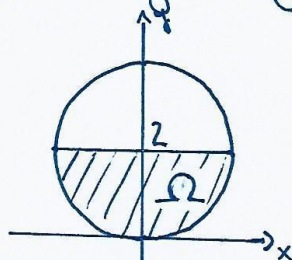
$$= \iint_{\Omega} \sqrt{1 + \frac{9x^2 + 9y^2}{3x^2 + 3y^2}} dx dy = \iint_{\Omega} \sqrt{1 + 3} dx dy =$$

$$= 2 \cdot \iint_{\Omega} dx dy = 2 \cdot S(\Omega) = 2 \cdot \pi \cdot 1^2 = \underline{\underline{2\pi}}$$

Skalární momenty

⑤

$$x^2 + y^2 \leq 4y, \quad y \leq 2, \quad \sigma(x, y) = 1, \quad S_x = ?$$

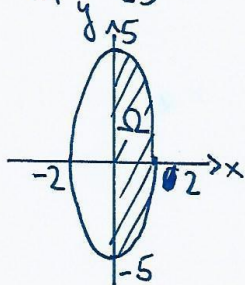


$$\begin{aligned} x &= \rho \cdot \cos \varphi & 0 \leq \rho \leq 2 \\ y &= \rho \cdot \sin \varphi + 2 & \pi \leq \varphi \leq 2\pi \\ J &= \rho \end{aligned}$$

$$\begin{aligned} S_x &= \iint_{\Omega} y \cdot \sigma(x, y) \cdot dx dy = \iint_{\Omega} y \cdot dx dy = \iint_{\Omega} (\rho \cdot \sin \varphi + 2) \cdot \rho \cdot d\rho d\varphi = \\ &= \int_0^2 d\rho \int_{\pi}^{2\pi} (\rho^2 \sin \varphi + 2\rho) d\varphi = \int_0^2 \left[-\frac{1}{2} \rho^2 \cos \varphi + 2\rho \varphi \right]_{\pi}^{2\pi} d\rho = \\ &= \int_0^2 (-\rho^2 + 4\pi\rho - \rho^2 - 2\pi\rho) d\rho = \int_0^2 (2\pi\rho - 2\rho^2) d\rho = \left[\pi\rho^2 - \frac{2\rho^3}{3} \right]_0^2 = \\ &= \underline{\underline{4\pi - \frac{16}{3}}} \end{aligned}$$

$$\textcircled{6} \quad 25x^2 + 4y^2 \leq 100, \quad x \geq 0, \quad S_y = ? \quad (\sigma(x, y) = 1)$$

$$\frac{x^2}{4} + \frac{y^2}{25} \leq 1 \quad \text{elipsa} \quad a=2, b=5$$



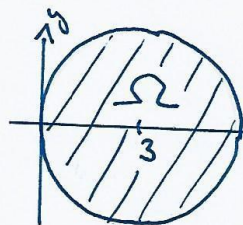
$$\begin{aligned} x &= 2 \cdot \rho \cdot \cos \varphi & 0 \leq \rho \leq 1 \\ y &= 5 \cdot \rho \cdot \sin \varphi & -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \\ J &= 10\rho \end{aligned}$$

$$S_y = \iint_{\Omega} x \cdot \sigma(x, y) \cdot dx dy =$$

$$\begin{aligned} &= \iint_{\Omega} x \cdot dx dy = \iint_{\Omega^*} 2 \cdot \rho \cdot \cos \varphi \cdot 10\rho \cdot d\rho d\varphi = \int_0^1 20\rho^2 \cdot d\rho \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \cdot d\varphi = \\ &= \left[\frac{20\rho^3}{3} \right]_0^1 \cdot \left[\sin \varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{20}{3} \cdot (1+1) = \underline{\underline{\frac{40}{3}}} \end{aligned}$$

Momenty setrvačnosti

- ⑦ Moment setrvačnosti homogenního kruhu vzhledem k jeho těžišti. ($r=3$) $\rho(x,y)=\rho$



$$x = \rho \cdot \cos \varphi + 3 \quad 0 \leq \rho \leq 3$$

$$y = \rho \cdot \sin \varphi$$

$$\rho = \rho$$

$$0 \leq \varphi \leq 2\pi$$

$$S_y = \iint_{\Omega} x^2 \rho(x,y) dx dy =$$

$$= \rho \cdot \iint_{\Omega} x^2 dx dy = \rho \cdot \iint_{\Omega} (\rho \cdot \cos \varphi + 3)^2 \cdot \rho \cdot d\rho d\varphi =$$

$$= \rho \cdot \iint_{\Omega} (\rho^3 \cos^2 \varphi + 6\rho^2 \cos \varphi + 9\rho) \cdot d\rho d\varphi =$$

$$= \rho \cdot \int_0^3 d\rho \int_0^{2\pi} \left(\frac{1}{2}\rho^3 + \frac{1}{2}\rho^3 \cos 2\varphi + 6\rho^2 \cos \varphi + 9\rho \right) \cdot d\varphi =$$

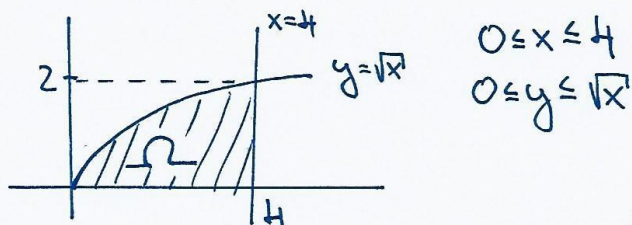
$$= \rho \cdot \int_0^3 \left[\frac{1}{2}\rho^3 \varphi + \frac{1}{4}\rho^3 \sin 2\varphi + 6\rho^2 \sin \varphi + 9\rho \varphi \right]_0^{2\pi} d\rho =$$

$$= \rho \cdot \int_0^3 (\pi \rho^3 + 18\pi \rho) \cdot d\rho = \rho \cdot \left[\frac{\pi \rho^4}{4} + 9\pi \rho^2 \right]_0^3 =$$

$$= \rho \cdot \left(\frac{81\pi}{4} + 81\pi \right) = \underline{\underline{\frac{405\pi \rho}{4}}}$$

Těžiště

⑧ $y = \sqrt{x}$, $y = 0$, $x = 4$, $\rho(x, y) = 1$, $T = ?$



$$m = \iint_{\Omega} \rho(x, y) dx dy = \iint_{\Omega} dx dy = \int_0^4 dx \int_0^{\sqrt{x}} dy = \int_0^4 [y]_0^{\sqrt{x}} dx = \int_0^4 \sqrt{x} dx =$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} \right]_0^4 = \frac{2 \cdot 4^{\frac{3}{2}}}{3} = \frac{16}{3}$$

$$S_x = \iint_{\Omega} y \cdot \rho(x, y) dx dy = \iint_{\Omega} y dx dy = \int_0^4 dx \int_0^{\sqrt{x}} y dy = \int_0^4 \left[\frac{y^2}{2} \right]_0^{\sqrt{x}} dx =$$

$$= \int_0^4 \frac{x}{2} dx = \left[\frac{x^2}{4} \right]_0^4 = 4$$

$$S_y = \iint_{\Omega} x \cdot \rho(x, y) dx dy = \iint_{\Omega} x dx dy = \int_0^4 dx \int_0^{\sqrt{x}} x dy = \int_0^4 [x \cdot y]_0^{\sqrt{x}} dx =$$

$$= \int_0^4 x \cdot \sqrt{x} dx = \int_0^4 x^{\frac{3}{2}} dx = \left[\frac{2x^{\frac{5}{2}}}{5} \right]_0^4 = \frac{2 \cdot 4^{\frac{5}{2}}}{5} = \frac{64}{5}$$

$$x_T = \frac{S_y}{m} = \frac{\frac{64}{5}}{\frac{16}{3}} = \frac{64}{5} \cdot \frac{3}{16} = \frac{12}{5}$$

$$y_T = \frac{S_x}{m} = \frac{4}{\frac{16}{3}} = 4 \cdot \frac{3}{16} = \frac{3}{4}$$

$$\underline{\underline{T = \left[\frac{12}{5}, \frac{3}{4} \right] = [2,4; 0,75] \in \Omega \quad \ddot{~}}}$$